

ESPECIALIZACIÓN EN ESTRUCTURAS ANÁLISIS ESTRUCTURAL AVANZADO

ELEMENTOS UNIDIMENSIONALES CASOS ESPECIALES

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ELEMENTOS UNIDIMENSIONALES

CASOS ESPECIALES

1. Subestructuración.
2. Igualación de GDL.
3. Condensación de GDL.
4. Articulaciones.
5. Nudos rígidos.



1. SUBESTRUCTURACIÓN

$$\{R\} + \{P\} = [K]\{D\} + \{EMP\}$$

$$\begin{Bmatrix} R_a \\ R_b \end{Bmatrix} + \begin{Bmatrix} P_a \\ P_b \end{Bmatrix} = \begin{bmatrix} K_{aa} & K_{ab} \\ K_{ba} & K_{bb} \end{bmatrix} \begin{Bmatrix} D_a \\ D_b \end{Bmatrix} + \begin{Bmatrix} EMP_a \\ EMP_b \end{Bmatrix}$$

$$\begin{Bmatrix} R_a = \{0\} \\ R_b \end{Bmatrix} + \begin{Bmatrix} P_a \\ P_b \end{Bmatrix} = \begin{bmatrix} K_{aa} & K_{ab} \\ K_{ba} & K_{bb} \end{bmatrix} \begin{Bmatrix} D_a \\ D_b = \{0\} \end{Bmatrix} + \begin{Bmatrix} EMP_a \\ EMP_b \end{Bmatrix}$$



1. SUBESTRUCTURACIÓN

$$\{R\} + \{P\} = [K]\{D\} + \{EMP\}$$

$$\begin{Bmatrix} \{0\} \\ \{R_b\} \end{Bmatrix} + \begin{Bmatrix} \{P_a\} \\ \{P_b\} \end{Bmatrix} = \begin{bmatrix} K_{aa} \\ K_{ba} \end{bmatrix} \{D_a\} + \begin{Bmatrix} EMP_a \\ EMP_b \end{Bmatrix}$$

$$\{0\} + \{P_a\} = [K_{aa}]\{D_a\} + \{EMP_a\}$$

$$\{R_b\} + \{P_b\} = [K_{ba}]\{D_a\} + \{EMP_b\}$$

1. SUBESTRUCTURACIÓN

$$\{P_a\} = [K_{aa}]\{D_a\} + \{EMP_a\}$$

$$[K_{aa}]\{D_a\} = \{P_a\} - \{EMP_a\}$$

$$[K_{aa}]^{-1}[K_{aa}]\{D_a\} = [K_{aa}]^{-1}(\{P_a\} - \{EMP_a\})$$

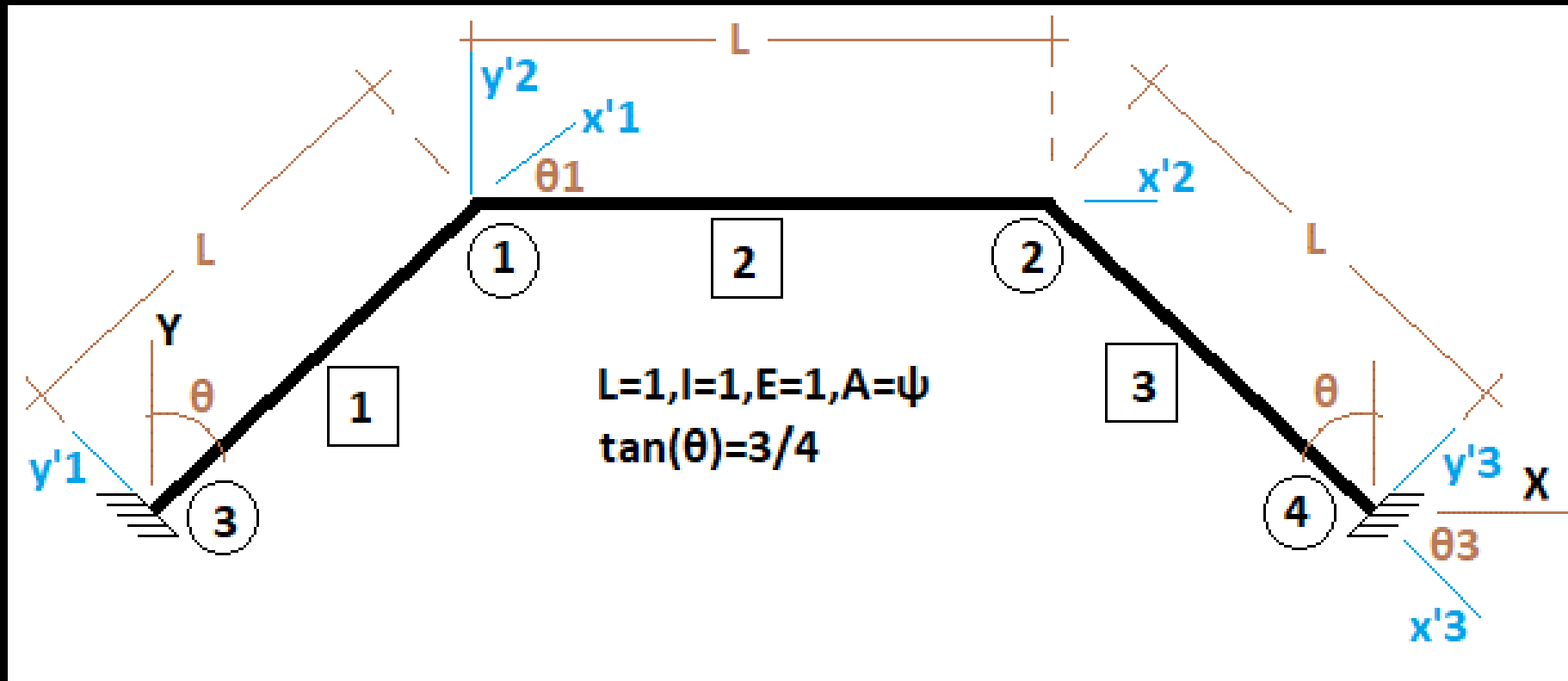
$$\{D_a\} = [K_{aa}]^{-1}(\{P_a\} - \{EMP_a\})$$

$$\{R_b\} = [K_{ba}]\{D_a\} + \{EMP_b\} - \{P_b\}$$



2. IGUALACIÓN DE GRADOS DE LIBERTAD

$$[A]\{U\} = \{0\}$$



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$$[A]\{U\} = \{0\}$$

$$\begin{bmatrix} 3 & 4 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 3 & -4 & 0 \end{bmatrix} \cdot \begin{bmatrix} ux1 \\ uy1 \\ \theta x1 \\ ux2 \\ uy2 \\ \theta x2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[A_d \quad A_i] \begin{Bmatrix} U_i \\ U_d \end{Bmatrix} = \{0\}$$

$$\left[\begin{bmatrix} 3 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 3 & -4 \end{bmatrix} \right] \cdot \begin{bmatrix} ux1 \\ \theta x1 \\ \theta x2 \\ uy1 \\ ux2 \\ uy2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\{U_d\} = -[A_d]^{-1}[A_i]\{U_i\} = [R_o]\{U_i\}$$

$$\{U\} = \begin{Bmatrix} U_i \\ U_d \end{Bmatrix} = \begin{bmatrix} I \\ R \end{bmatrix} \{U_i\} = [R]\{U_i\}$$



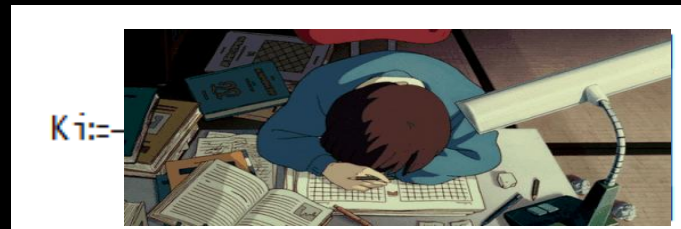
$$R_o := \begin{bmatrix} -\frac{3}{4} & 0 & 0 \\ 1 & 0 & 0 \\ -\frac{3}{4} & 0 & 0 \end{bmatrix} \quad R := \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{4} & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{3}{4} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. IGUALACIÓN DE GRADOS DE LIBERTAD

$$[F] = \{K\}\{U\}$$

$$F = \begin{bmatrix} \frac{2 \cdot (17 \cdot \psi + 96)}{25} & \frac{12 \cdot (\psi - 12)}{25} & \frac{24}{5} & -\psi & 0 & 0 \\ \frac{12 \cdot (\psi - 12)}{25} & \frac{8 \cdot (2 \cdot \psi + 51)}{25} & \frac{12}{5} & 0 & -12 & 6 \\ \frac{24}{5} & \frac{12}{5} & 8 & 0 & -6 & 2 \\ -\psi & 0 & 0 & \frac{2 \cdot (17 \cdot \psi + 96)}{25} & \frac{12 \cdot (12 - \psi)}{25} & \frac{24}{5} \\ 0 & -12 & -6 & \frac{12 \cdot (12 - \psi)}{25} & \frac{8 \cdot (2 \cdot \psi + 51)}{25} & \frac{12}{5} \\ 0 & 6 & 2 & \frac{24}{5} & \frac{12}{5} & 8 \end{bmatrix} \cdot \begin{bmatrix} ux1 \\ \theta x1 \\ \theta x2 \\ uy1 \\ ux2 \\ uy2 \end{bmatrix}$$

$$\{F_i\} = [R]^T \{F\} = [R]^T (\{K\}\{U\}) = [R]^T (\{K\}([R]\{U_i\})) = ([R]^T \{K\}[R])\{U_i\} = [K_i]\{U_i\}$$



3. CONDENSACIÓN DE GDL

$$[F] = [K]\{U\}$$

$$\{U\} = \begin{Bmatrix} U_c \\ U_o \end{Bmatrix}$$

$$[K] = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

$$[F] = \begin{Bmatrix} F_c \\ 0 \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} U_c \\ U_o \end{Bmatrix}$$

$$[F_c] = [K_{11}]\{U_c\} + [K_{12}]\{U_o\}$$

$$[0] = [K_{21}]\{U_c\} + [K_{22}]\{U_o\}$$

$$\{U_o\} = -[K_{22}]^{-1}[K_{21}]\{U_c\}$$

$$[F_c] = ([K_{11}] - [K_{12}][K_{22}]^{-1}[K_{21}])\{U_c\}$$

$$[K_c] = ([K_{11}] - [K_{12}][K_{22}]^{-1}[K_{21}])$$

$$[F_c] = [K_c]\{U_c\}$$

$$\{U_c\} = [K_c]^{-1}[F_c]$$

$$\begin{bmatrix} F_{x1} \\ M1=0 \\ M2=0 \end{bmatrix} = \frac{1}{25} \cdot \begin{bmatrix} 1612.5 & -37.5 & -37.5 \\ -37.5 & 200 & 50 \\ -37.5 & 50 & 200 \end{bmatrix} \cdot \begin{bmatrix} u_{x1} \\ \theta_{z1} \\ \theta_{z2} \end{bmatrix}$$

$$[K_c] = ?$$

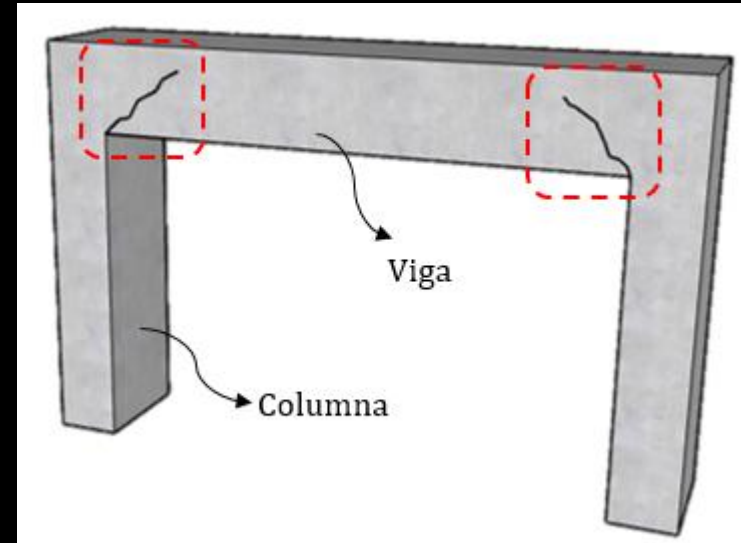
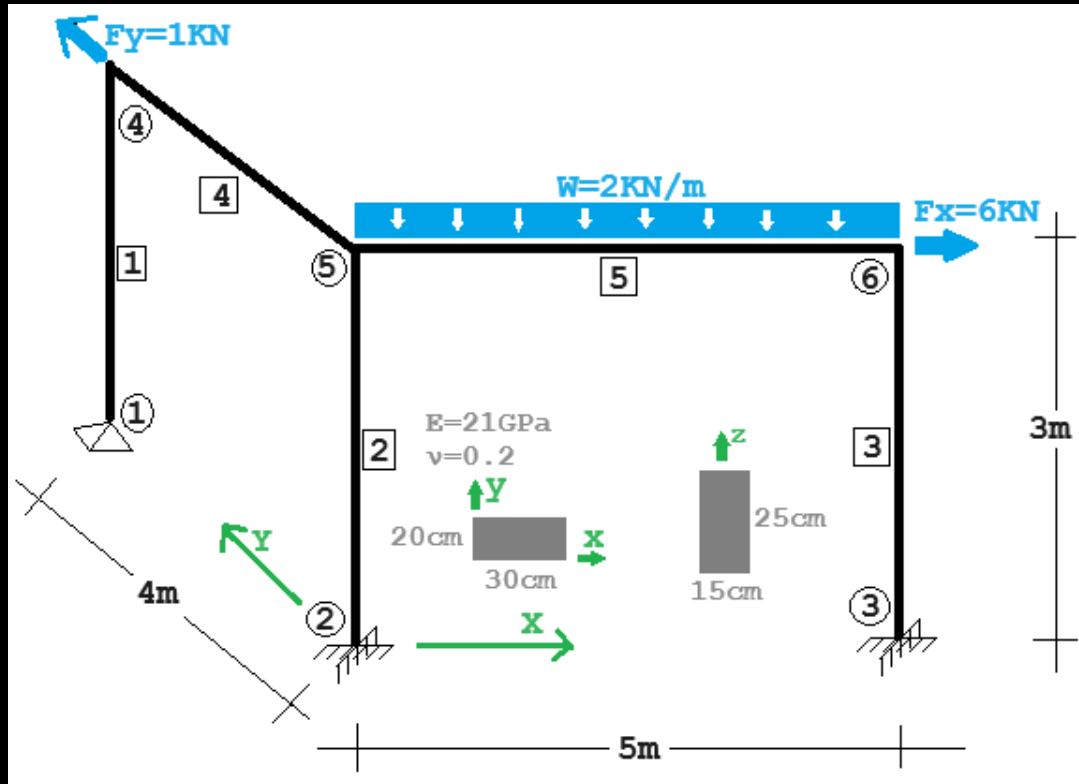
$$\{U_c\} = ?$$

$$\{U_o\} = ?$$

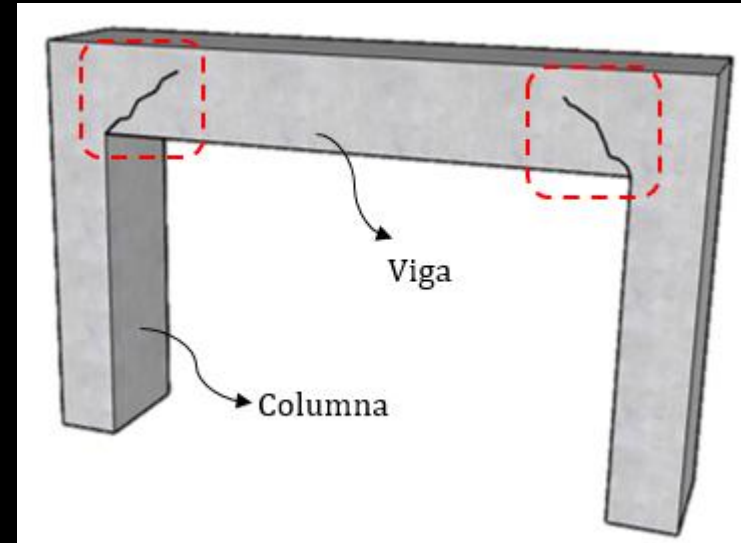
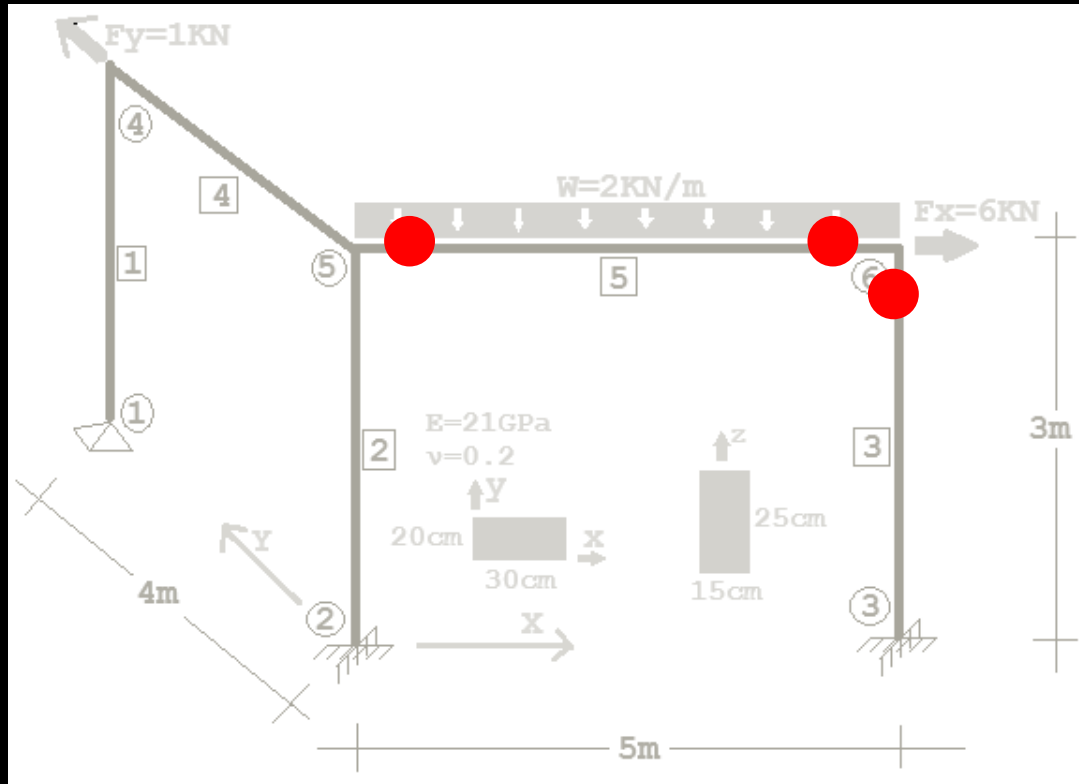
$$[F_c] = ?$$



4. ARTICULACIONES



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4. ARTICULACIONES

Condensar los gdl a articular:

$$[f'] = [k']\{u'\}$$

$$\{u'\} = \begin{Bmatrix} u'_c \\ u'_o \end{Bmatrix}$$

$$[k'] = \begin{bmatrix} k'_{11} & k'_{12} \\ k'_{21} & k'_{22} \end{bmatrix}$$

$$[f'] = \begin{Bmatrix} F'_c \\ 0 \end{Bmatrix} = \begin{bmatrix} k'_{11} & k'_{12} \\ k'_{21} & k'_{22} \end{bmatrix} \begin{Bmatrix} u'_c \\ u'_o \end{Bmatrix}$$

$$[f'_c] = [k'_{11}]\{u'_c\} + [k'_{12}]\{u'_o\}$$

$$[0] = [k'_{21}]\{u'_c\} + [k'_{22}]\{u'_o\}$$

$$\{u'_o\} = -[k'_{22}]^{-1}[k'_{21}]\{u'_c\}$$

$$[f_c] = ([k'_{11}] - [k'_{12}][k'_{22}]^{-1}[k'_{21}])\{u'_c\}$$

$$[k'_c] = ([k'_{11}] - [k'_{12}][k'_{22}]^{-1}[k'_{21}])$$

$$[f'_c] = [k'_c]\{u'_c\}$$

$[k''_c]$ = adecuar $[k'_c]$ con filas y columnas de ceros en los gdl articulados



4. ARTICULACIONES

Condensar un solo (i) gdl a articular:

$$[f'] = [k']\{u'\}$$

$$\{u'\} = \begin{Bmatrix} u'_c \\ u'_0 \end{Bmatrix}$$

$$[k'] = \begin{bmatrix} k'_{11} & k'_{12} \\ k'_{21} & k'_{22} \end{bmatrix}$$

$$[k'_c] = ([k'_{11}] - [k'_{12}][k'_{22}]^{-1}[k'_{21}])$$

$$[k'_c] = ([k'] - [k'_{columna\ i}][k'_{ii}]^{-1}[k'_{fila\ i}])$$

$$[f'_c] = [k'_c]\{u'_c\}$$

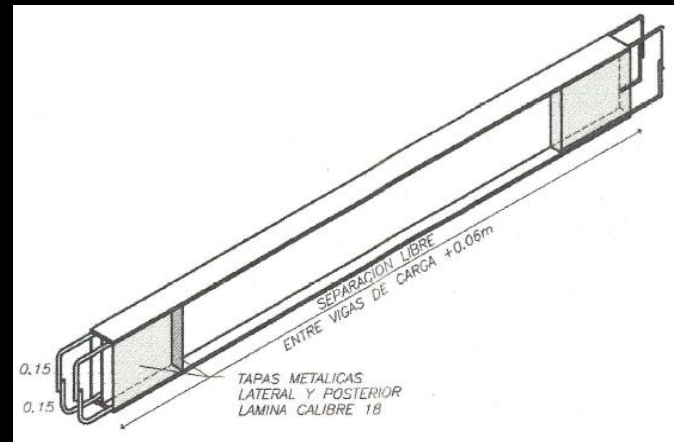
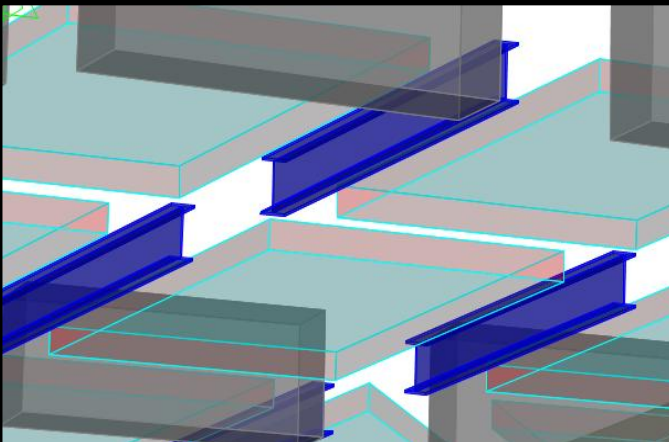
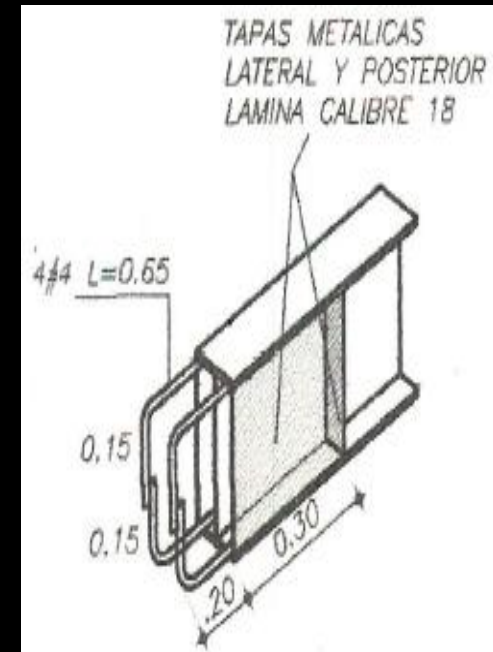
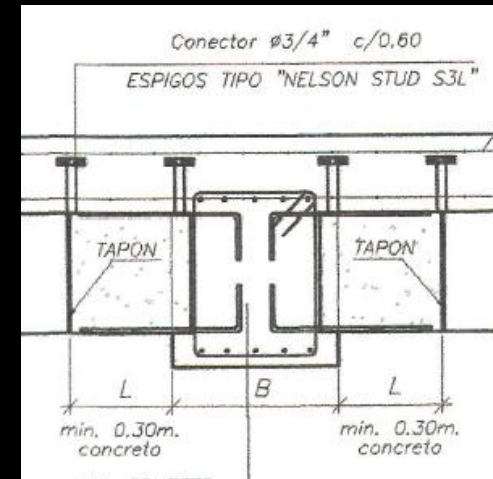
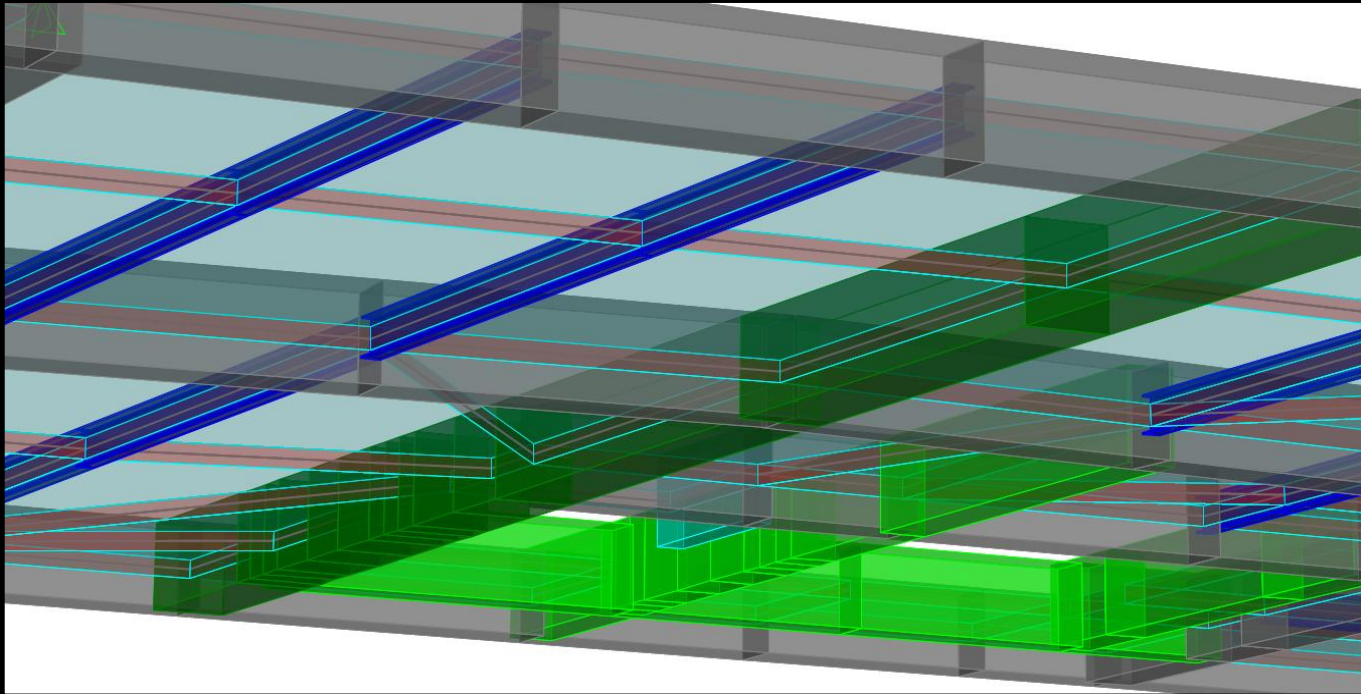
$\{emp'\}$: El vector de empotramiento debe adecuarse a las condiciones de apoyo articulado



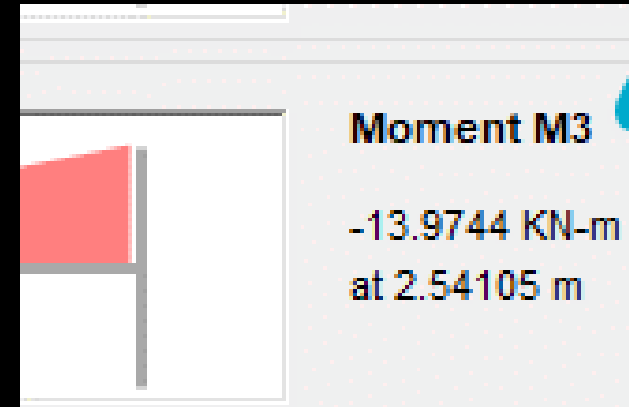
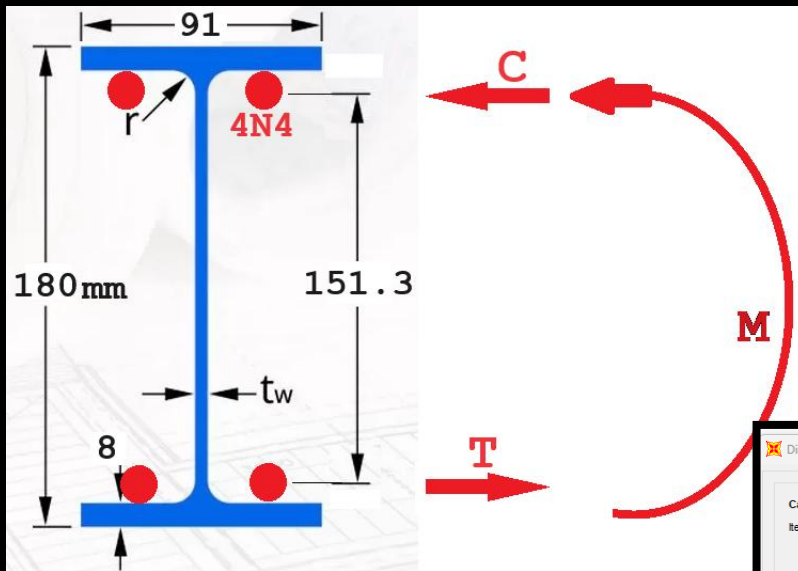
4. ARTICULACIONES



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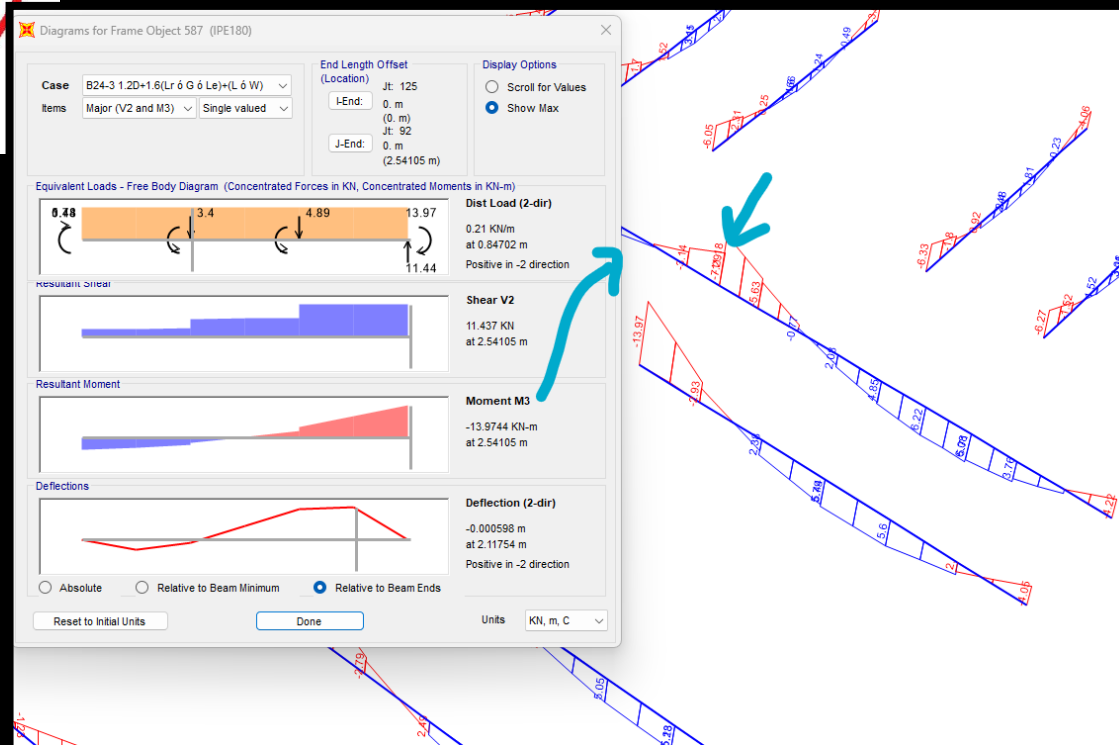


4. ARTICULACIONES

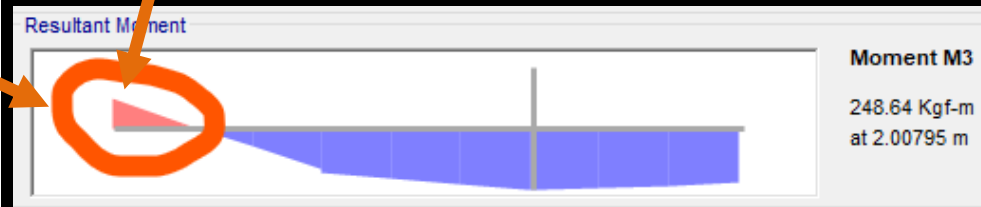
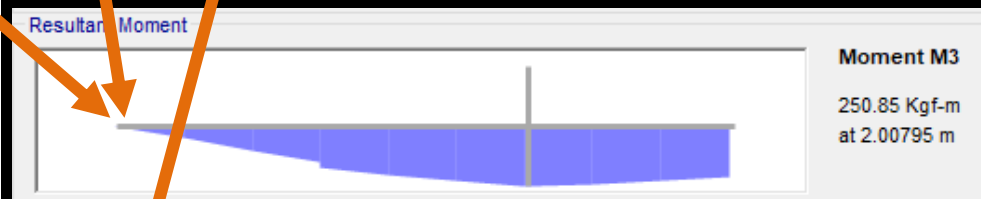
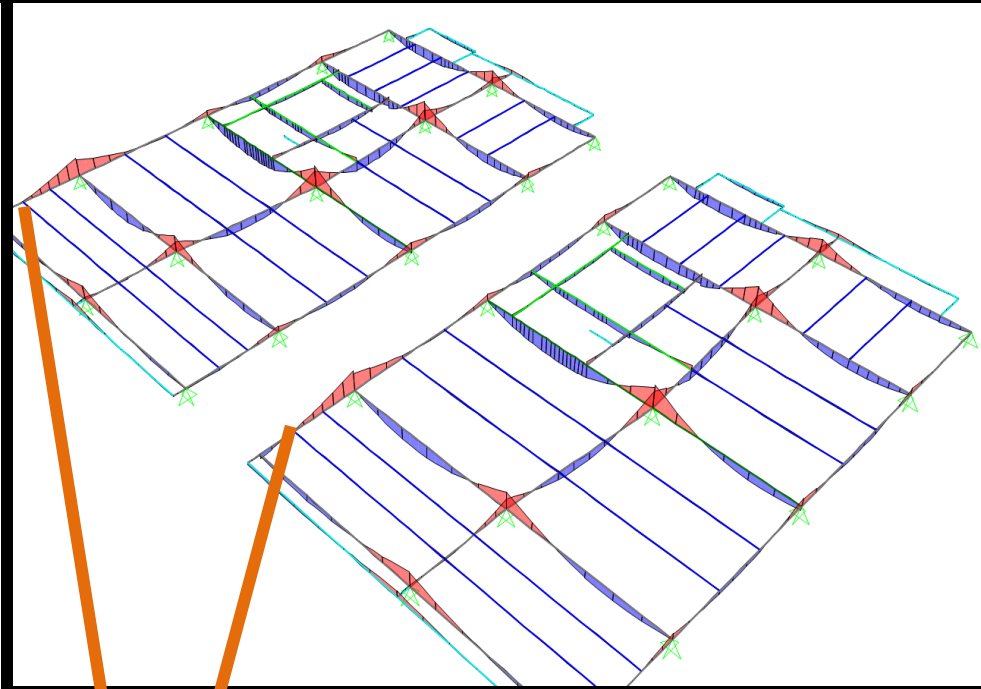
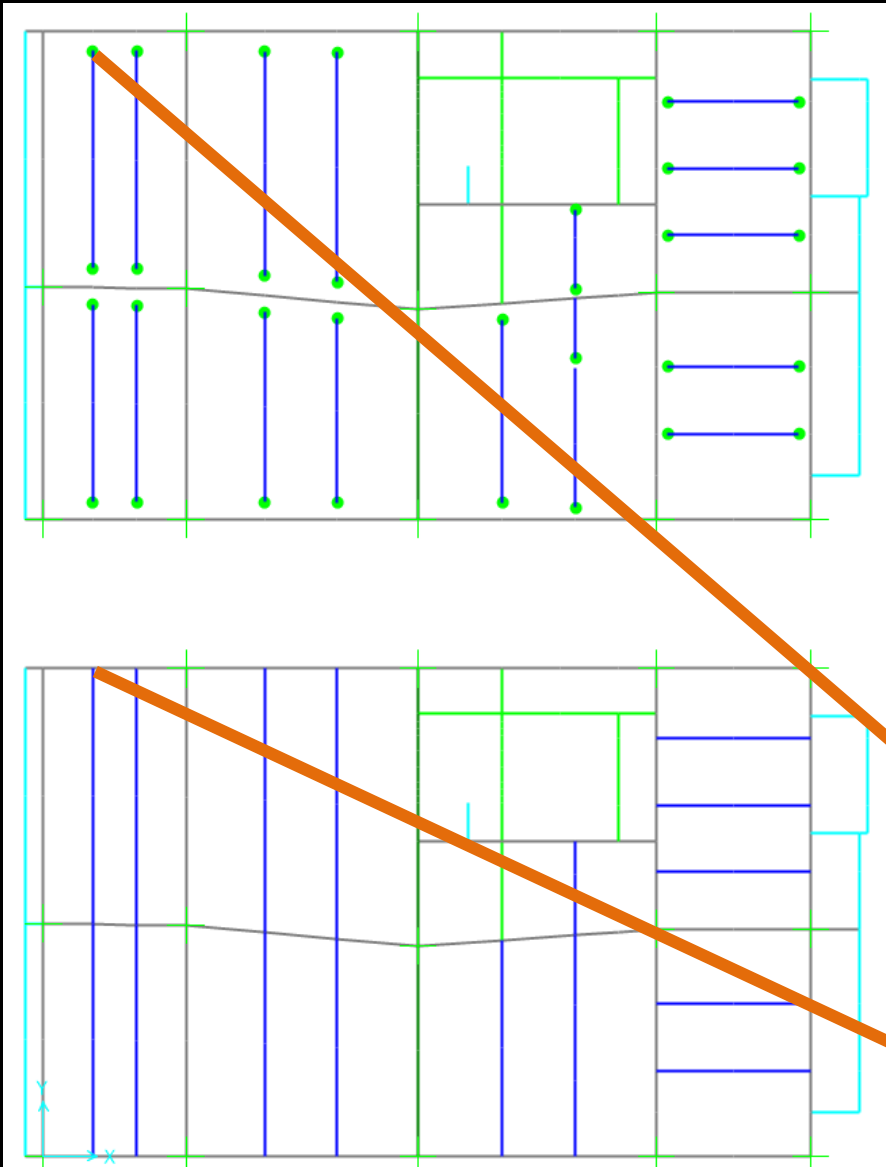


Para barras $f_y=420$ Mpa:
Momento flector nominal
máximo: 8.2 kN.m

Momento flector último
máximo: 7.4 kN.m



4. ARTICULACIONES



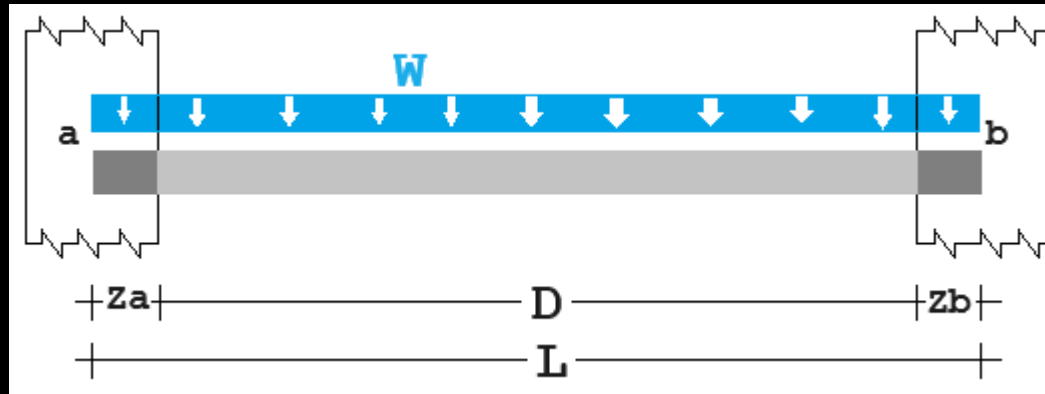
5. NUDOS RÍGIDOS



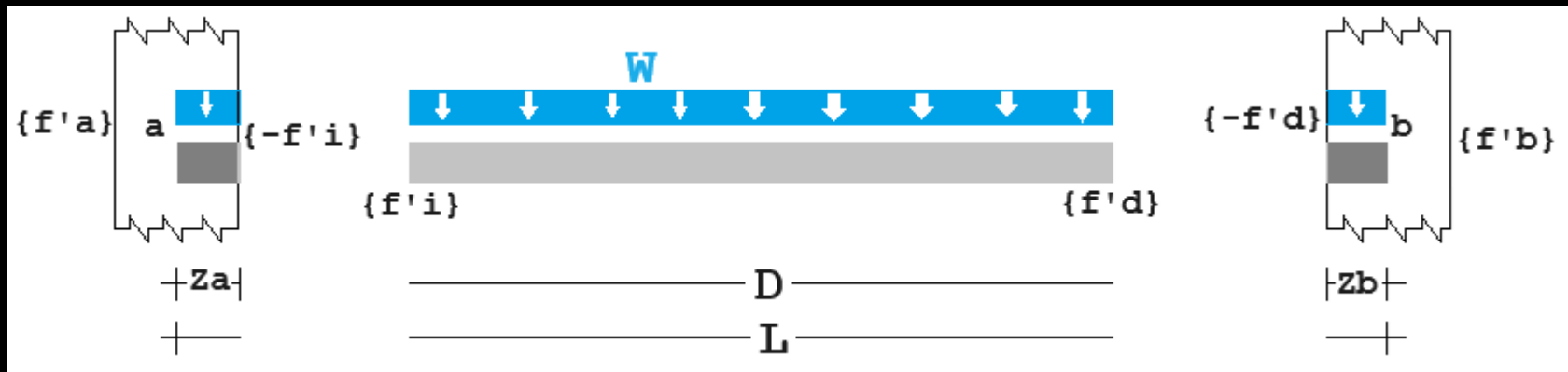
5. NUDOS RÍGIDOS



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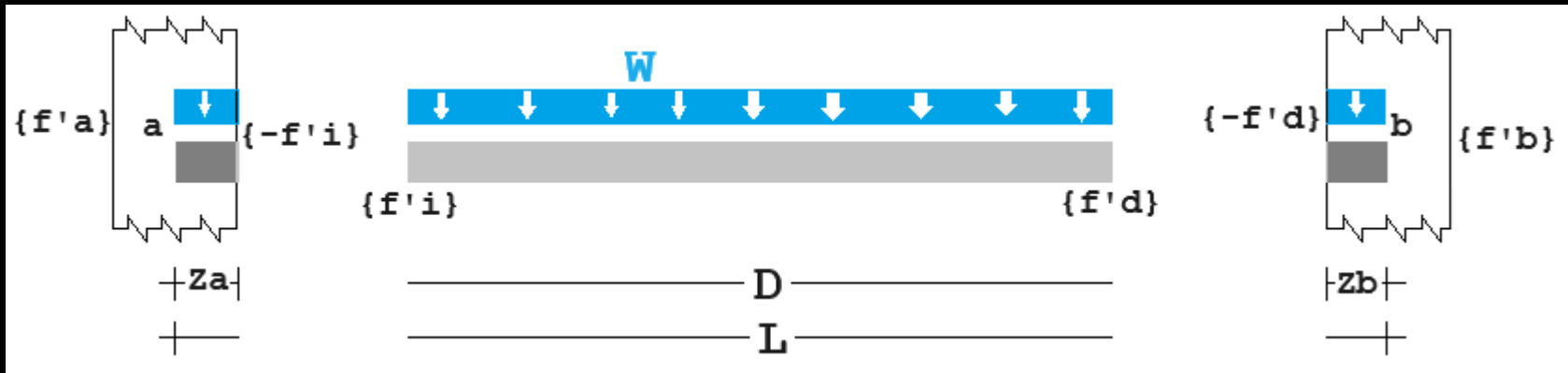


Diagramas de cuerpo libre:



5. NUDOS RÍGIDOS

Diagramas de cuerpo libre:



$$\{p'_L\} = \begin{Bmatrix} \{f'i\} \\ \{f'd\} \end{Bmatrix}$$

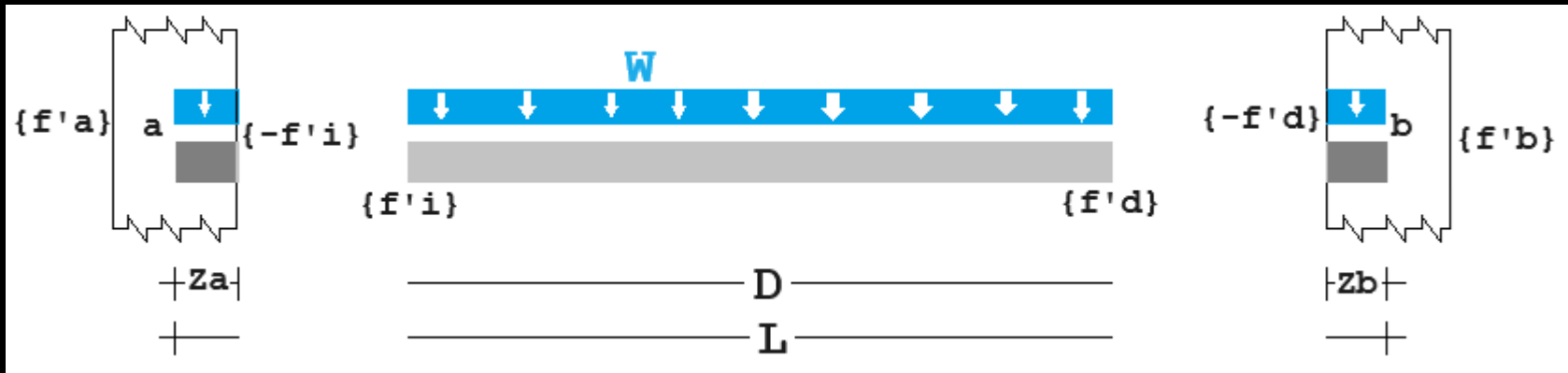
$$\{f'\} = [T]\{p'_L\}$$

$$\{f'\} = \begin{Bmatrix} x'_a \\ y'_a \\ mz'_a \\ x'_b \\ y'_b \\ mz'_b \end{Bmatrix} = \begin{Bmatrix} x'_i \\ y'_i \\ mz'_i + Za * y'_i \\ x'_d \\ y'_d \\ md'_d - Zb * y'_d \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x'_i \\ y'_i \\ mz'_i \\ x'_d \\ y'_d \\ mz'_d \end{Bmatrix}$$



5. NUDOS RÍGIDOS

Diagramas de cuerpo libre:



$$\{p'_L\} = \begin{Bmatrix} \{f'i\} \\ \{f'd\} \end{Bmatrix}$$

$$\{p'_L\} = [k'_L]\{u'_L\}$$

$$\{f'\} = [T]\{p'_L\}$$

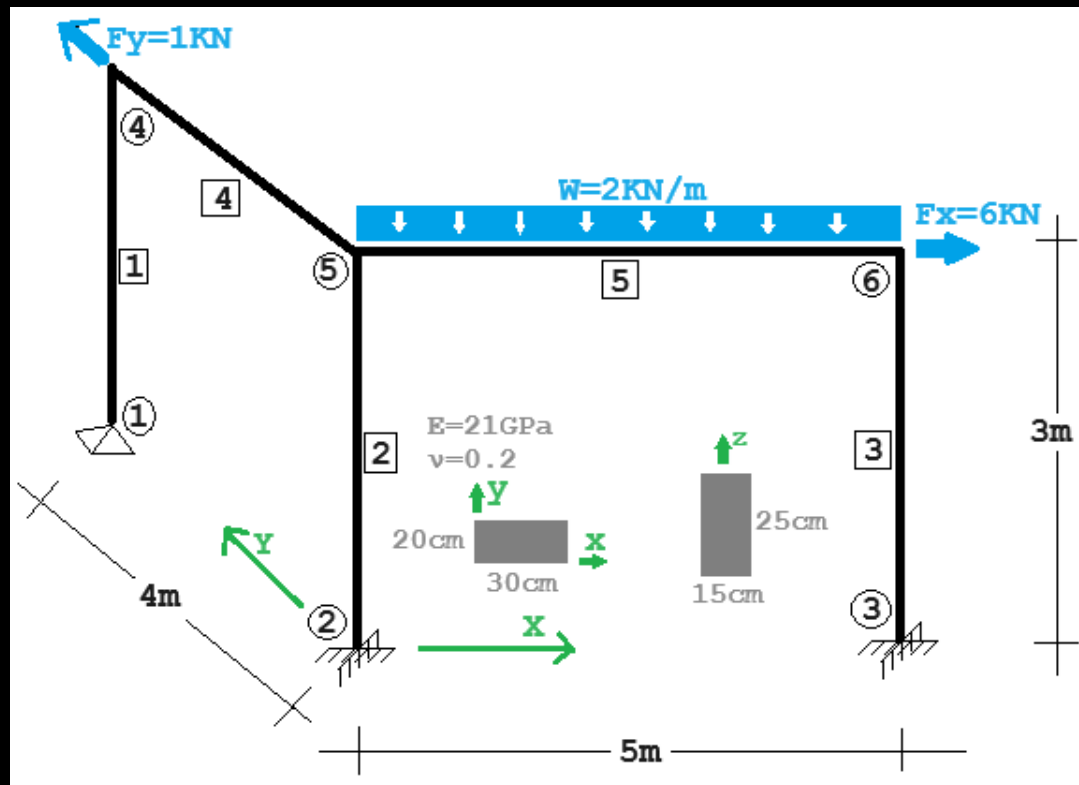
$$\{f'\} = [T][k'_L]\{u'_L\}$$

$$\{u'_L\} = [T]^T\{u'\}$$

$$\{f'\} = [T][k'_L][T]^T\{u'\}$$

$$\{f'\} = [k]\{u'\}$$

5. NUDOS RÍGIDOS



Gracias

Michel Bolaños Guerrero, PhD

Créditos a:

<https://openai.com/dall-e-2>

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